Effects of colored noise on stochastic resonance in a bistable system subject to multiplicative and additive noise

Ya Jia,[1,*] Xiao-ping Zheng,[1] Xiang-ming Hu,[1] and Jia-rong Li[2]
[1]*Department of Physics, Huazhong Normal University, Wuhan 430079, China*[†]
[2]*Institute of Particle Physics, Huazhong Normal University, Wuhan 430079, China*
(Received 18 October 2000; published 23 February 2001)

The effects of colored noise on stochastic resonance (SR) in a bistable system driven by multiplicative colored noise and additive white noise and a periodic signal are studied by using the unified colored noise approximation and the theory of signal-to-noise ratio (SNR) in the adiabatic limit. In the case of no correlations between noises, there is an optimal noise intensities ratio R at which SNR is a maximum that identifies the characteristics of the SR when the correlation time τ of the multiplicative colored noise is small. However, when τ is increased, a second optimal value of R appears, and two peaks appear in the SNR simultaneously. In the case of correlations between noises, the SNR is not only dependent on the correlation time τ, but also on the intensity of correlations between noises. Moreover, the double peak phenomenon can also appear as τ is increased in certain situations.

DOI: 10.1103/PhysRevE.63.031107 PACS number(s): 05.40.−a

I. INTRODUCTION

Since stochastic resonance (SR) was proposed to explain the periodic recurrences of the earth's ice ages [1,2], the phenomenon has been extensively studied from both the theoretical and experimental points of view [3–7]. SR is a name coined for the rather counterintuitive fact that the response of a nonlinear system to a periodic signal may be enhanced through the addition of an optimal amount of noise.

There have been many theoretical developments of SR in conventional bistable systems [8–20]. McNamara, Wiesenfeld and Roy [8,9] have suggested a master equation for the populations in two stable states. They considered the signal-to-noise ratio (SNR), i.e., the ratio of the δ peak height in the power spectrum to the noise background as a probe of the SR effect. Zhou, Moss, and Jung [14] have suggested the escape time distribution to describe SR. Jung and Hänggi [15] described SR within the framework of nonstationary stochastic processes without restriction to small driving amplitudes or frequencies, where they presented power spectral densities and signal amplification as measures of SR.

The largest amount of work about the SR phenomenon has referred to the consideration of systems with just one noise source. However, many physical systems require considering various noise sources. Moreover, in certain situations noises may be correlated with each other. Recently, the SR phenomenon in a conventional bistable system under the simultaneous action of multiplicative and additive noise and a periodic signal has been discussed by using the theory of SNR in Ref. [21]. It should be pointed out that the multiplicative noise and additive noise are all assumed as white Gaussian noise in Ref. [21]. However, more realistic models of physical systems require considering the case of colored noise, especially the stochastic system driven by white Gaussain noise and colored noise. This situation is generic for a variety of physical situations, for example, the biological transport that works in the presence of white thermal noise and correlated random noise of biological origin, and the dynamics of a dye laser, etc. Therefore, it is very important to study the effects of colored noise on the SR phenomenon of nonlinear systems. In this paper, we will use the theory of SNR proposed by McNamara and Wiesenfeld [9] to study the effects of colored noise on the SR in conventional bistable systems under the simultaneous action of a multiplicative colored noise and an additive white noise and a periodic signal.

According to the theory of Ref. [9], the bistable case is reduced to a two-state system, characterized by the occupation probabilities $n_{\pm}=\mathrm{prob}(x=x_{\pm})$ of both stable states $x_{\pm}$. The master equation for these occupation probabilities is

$$\begin{aligned}\dot{n}_{+}=-\dot{n}_{-}&=W_{-}(t)n_{-}-W_{+}(t)n_{+}\\&=W_{-}(t)-[W_{-}(t)+W_{+}(t)]n_{+},\end{aligned}\tag{1}$$

where $W_{\pm}$ is the transition rate out of stable states $x_{\pm}$. To obtain an expression of SNR in terms of the output signal power spectrum, the key problem is to calculate the transition rate. It must be stressed that the expression for the transition rate would be valid only in the *adiabatic limit*, so this theory of SNR is also called the adiabatic approximation. In order to keep our results' validity throughout this paper, we will also restrict ourselves to the case of the adiabatic limit. On the other hand, different theories have been used to deal with the colored noise problem, for instance, the conventional small-τ theory [22], the functional calculus theory of Fox [23], the decoupling theory (often called the Hänggi ansatz) [24], the unified colored noise approximation

*Electronic address: jiay@phy.ccnu.edu.cn
[†]Mailing address.

(UCNA) [25], etc. Here we will apply the UCNA to study the effects of colored noise on the SNR. Because the UCNA is valid for both small and large correlation times of the colored noise [25], our results are valid in a large region of the τ value. In Sec. II the general theory of nonlinear system driven by multiplicative colored noise and additive white noise is given by using the UCNA. Considering conventional bistable system, we study the effects of colored noise on the SNR for two cases: the case of no correlations between noises and the case of correlations between noises in Sec. III. We end with conclusions in Sec. IV.

II. GENERAL THEORY OF COLORED NOISE

Let us consider the overdamped motion of a Brownian particle in a potential $U_0(x)$ that has two stable states $x_\pm$ and an unstable state x_u, the stochastic system under the simultaneous action of multiplicative colored noise and additive white noise is described by the Langevin equation

$$\dot{x}=f(x)+g(x)\epsilon(t)+\eta(t), \tag{2}$$

where $f(x)=-U_0'(x)$ and

$$\langle\epsilon(t)\rangle=0,\quad \langle\epsilon(t)\epsilon(s)\rangle=\frac{Q}{\tau}\exp\left(\frac{|t-s|}{\tau}\right), \tag{3}$$

$$\langle\eta(t)\rangle=0,\quad \langle\eta(t)\eta(s)\rangle=2D\delta(t-s), \tag{4}$$

Q and τ denote the intensity and the correlation time of the multiplicative Gaussian colored noise $\epsilon(t)$, and D denotes the intensity of the additive Gaussian white noise $\eta(t)$.

The one-dimensional non-Markovian process (2) with Eqs. (3) and (4) is stochastically equivalent to two-dimensional Markovian processes

$$\dot{x}=f(x)+g(x)\epsilon(t)+\eta(t), \tag{5}$$

$$\dot{\epsilon}=-\frac{1}{\tau}\epsilon+\frac{1}{\tau}\xi(t), \tag{6}$$

where $\xi(t)$ is another Gaussian white noise with $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(s)\rangle=2Q\delta(t-s)$. Applying the UCNA to two-dimensional Markovian processes (5) and (6), one can obtain the following one-dimensional Markovian approximation [25,26]:

$$\dot{x}=C^{-1}(x,\tau)[f(x)+g(x)\xi(t)+\eta(t)], \tag{7}$$

where

$$C(x,\tau)=1-\tau\left(f'(x)-\frac{g'(x)}{g(x)}f(x)\right), \tag{8}$$

the prime denotes differentiation with respect to *x*. Then the Fokker-Planck equation corresponding to Eq. (7) can be read as

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}A(x,\tau)P(x,t)+\frac{\partial^2}{\partial x^2}B(x,\tau)P(x,t) \tag{9}$$

with

$$A(x,\tau)=\frac{f(x)}{C(x,\tau)}+\frac{K'(x)}{C^2(x,\tau)}-\frac{C'(x,\tau)K(x)}{C^3(x,\tau)},$$

$$B(x,\tau)=\frac{K(x)}{C^2(x,\tau)}, \tag{10}$$

where the state-dependent function $K(x)$ is also dependent on either correlations or no correlations between the noises $\xi(t)$ and $\eta(t)$. When $\xi(t)$ is correlated with $\eta(t)$ according to $\langle\eta(t)\xi(s)\rangle=\langle\xi(t)\eta(s)\rangle=2\lambda\sqrt{QD}\delta(t-s)$, where λ denotes the strength of the correlations between $\xi(t)$ and $\eta(t)$, and $|\lambda|\leqslant 1$, then $K(x)$ becomes

$$\begin{aligned}K(x,\lambda)&=Qg^2(x)+2\lambda\sqrt{QD}g(x)+D\\&=D[Rg^2(x)+2\lambda\sqrt{R}g(x)+1],\end{aligned} \tag{11}$$

$R\equiv Q/D$ is the ratio of noise intensities. The stationary probability distribution $P_s(x)$ of Eq. (9) is

$$P_s(x)=\frac{N}{|D_{eff}(x,\tau,\lambda)|^{1/2}}\exp\left(-\frac{\Phi(x,\tau,\lambda)}{D}\right), \tag{12}$$

where the effective diffusion $D_{eff}(x,\tau,\lambda)$ and the generalized potential $\Phi(x,\tau,\lambda)$ are given by

$$D_{eff}(x,\tau,\lambda)=\frac{K(x,\lambda)}{C^2(x,\tau)},$$

$$\Phi(x,\tau,\lambda)=-D\int\frac{f(x)C(x,\tau)}{K(x,\lambda)}dx. \tag{13}$$

The mean first passage time (MFPT) $\mathcal{T}_\pm$ of the process $x(t)$ to reach the state $x_\mp$ with initial condition $x(t=0)=x_\pm$ is given by the Kramers time when $D\ll 1$ [27–30]

$$\begin{aligned}\mathcal{T}_\pm&=2\pi|U_0''(x_\pm)U_0''(x_u)|^{-1/2}\\&\quad\times\exp\left[\frac{\Phi(x_u,\tau,\lambda)-\Phi(x_\pm,\tau,\lambda)}{D}\right].\end{aligned} \tag{14}$$

Then, one can obtain the transition rates $W_\pm$ out of $x_\pm$ according to $W_\pm=\mathcal{T}_\pm^{-1}$ [9].

III. EFFECTS OF COLORED NOISE ON STOCHASTIC RESONANCE

Now consider $U_0(x)$ is a conventional symmetric bistable potential, and assume that the system Eq. (2) is driven by a periodic signal (or periodic forcing), then the dimensionless form of the Langevin equation for this system can be read as

$$\dot{x}=x-x^3+x\epsilon(t)+A\cos\Omega t+\eta(t), \tag{15}$$

where the statistical properties of the noises $\epsilon(t)$ and $\eta(t)$ are given by Eqs. (3) and (4), A is the amplitude, and Ω is the frequency of the periodic signal.

In the absence of the periodic signal, the deterministic potential of the bistable system is $U_0(x)=-x^2/2+x^4/4$, which has two stable states $x_-=-1$, $x_+=+1$ and an unstable state $x_u=0$. In the presence of the periodic signal, the potential of the system is modulated by the signal. However, it is assumed [9] that the signal amplitude is small enough (i.e., $A\ll1$) that, in the absence of any noise, it is insufficient to force a particle to move from one well to the other and it can be considered that $x_\pm=\pm1$ and $x_u=0$ are still the stable states and unstable state of the system. Moreover, the variation of the periodic signal is slow enough (i.e., $\Omega\ll1$ or in the adiabatic limit) that there is enough time to make the system reach local equilibrium in the period of $1/\Omega$. Therefore, one can obtain the quasi-steady-state distribution function of the system in the adiabatic limit. In order to discuss the effects of colored noise on the stochastic resonance, two cases of the correlation between noises will be considered.

A. The case of $\lambda=0$

When there is no correlation between $\xi(t)$ and $\eta(t)$ (i.e., $\lambda=0$) and $\Omega\ll1$ (the adiabatic limit), the quasi-steady-state distribution function $P_s(x,t)$ of system can be written as

$$P_s(x,t)=\frac{N}{|D_{eff}(x,\tau,\lambda=0,t)|^{1/2}}\exp\left[-\frac{\Phi(x,\tau,\lambda=0,t)}{D}\right], \tag{16}$$

and the generalized potential $\Phi(x,\tau,\lambda=0,t)$ can be expressed as

$$\begin{aligned}\Phi(x,\tau,\lambda=0,t)=&\frac{\tau}{2R}x^4-\frac{2\tau+2\tau R-R}{2R^2}x^2\\&+\frac{2\tau+(2\tau-1)R-R^2}{2R^3}\ln|Rx^2+1|\\&+\left(-\frac{\tau}{R}x+\frac{\tau-R\tau-R}{R\sqrt{R}}\arctan\sqrt{R}x\right)\\&\times A\cos\Omega t+O(A^2),\end{aligned} \tag{17}$$

where the signal amplitude is very small (i.e. $A\ll1$). The MFPT $\mathcal{T}_\pm$ of the process $x(t)$ to reach the state $x_\mp$ with initial condition $x(t=0)=x_\pm$ is given by Eq. (14), thus the transition rates $W_\pm$ out of $x_\pm$ are approximately

$$\begin{aligned}W_\pm\simeq&\frac{1}{\sqrt{2}\pi}\exp\Bigg\{-\frac{1}{D}\Bigg[-\frac{\tau}{2R}+\frac{2\tau+2R\tau-R}{2R^2}\\&-\frac{2\tau+(2\tau-1)R-R^2}{2R^3}\ln|R+1|\\&\pm\left(\frac{\tau}{R}-\frac{\tau-R\tau-R}{R\sqrt{R}}\arctan\sqrt{R}\right)A\cos\Omega t\Bigg]\Bigg\}.\end{aligned} \tag{18}$$

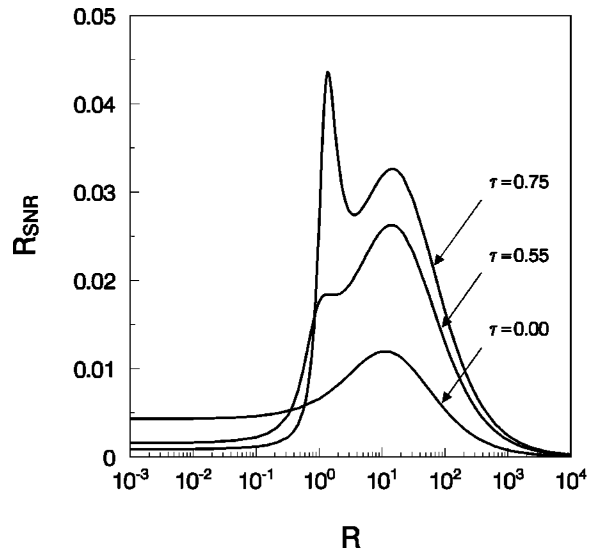

FIG. 1. SNR for the case of no correlations between noises as a function of the noise intensities ratio R for different values of the correlation time τ of the colored noise. $A=0.05$, $\Omega=0.001$, and $D=0.05$.

Within the framework of the theory of SNR [9], we can obtain the standard form of the signal-to-noise ratio R_{SNR} for the bistable system with no correlations between noises in terms of the output signal power spectrum,

$$\begin{aligned}R_{SNR}=&\frac{\pi W_0A^2}{4D^2}\left(\frac{\tau}{R}-\frac{\tau-R\tau-R}{R\sqrt{R}}\arctan\sqrt{R}\right)^2\\&\times\Bigg[1-\frac{W_0^2A^2}{2D^2(W_0^2+\Omega^2)}\\&\times\left(\frac{\tau}{R}-\frac{\tau-R\tau-R}{R\sqrt{R}}\arctan\sqrt{R}\right)^2\Bigg]^{-1},\end{aligned} \tag{19}$$

where

$$\begin{aligned}W_0=&\frac{\sqrt{2}}{\pi}\exp\Bigg\{-\frac{1}{D}\Bigg[-\frac{\tau}{2R}+\frac{2\tau+2R\tau-R}{2R^2}\\&-\frac{2\tau+(2\tau-1)R-R^2}{2R^3}\ln|R+1|\Bigg]\Bigg\}.\end{aligned} \tag{20}$$

By virtue of the expression [Eq. (19)] of SNR, the effects of colored noise on SNR can be discussed by numerical computation. In Fig. 1 we present the SNR as a function of the noise intensities ratio $R(\equiv Q/D)$ for different values of correlation time τ of the colored noise. When $\tau=0$ (the case of white noise) and τ is small, there is a maximum in the SNR at the moderate value of the noise intensities ratio R. It means that there is an optimal ratio of noise intensities at which the SNR of the system is a maximum (i.e., there is one peak) that identifies as characteristic of the SR phenomenon, we can call this phenomenon as *single stochastic resonance* (SSR). The largest amount of previous investigations about SR has referred to the SSR phenomenon. The peak of the SNR increases on increasing the correlation time. However, when the correlation time is increased, a second peak appears at a smaller value of the noise intensities ratio R, and the peak becomes high as the correlation time increases. It has been shown that, when the multiplicative noise is Gaussian colored noise, there can be two optimal values of the noise intensities ratio, that is, there are two peaks in the SNR at which the stochastic resonance occurs, so we may call this phenomenon as *double stochastic resonance* (DSR). Although a similar phenomenon has been shown in Refs. [9] and [21], where this phenomenon appears for a sufficiently low frequency of the input signal [9] and for the increasing amplitude of the input signal [21], respectively, yet for the increasing correlation time τ of the multiplicative colored noise here. Moreover, the first peak is very broad and low in Refs. [9,12]. Another interesting point here is that, when $R \to 0$, the SNR decreases but saturates to a plateau value; however, the signal-to-noise ratio will vanish when $R \to \infty$.

B. The case of $\lambda \neq 0$

In the presence of the correlations between $\xi(t)$ and $\eta(t)$ (i.e., $\lambda \neq 0$), when the signal frequency is very low ($\Omega \ll 1$) and the signal amplitude is very small ($A \ll 1$), the quasi-steady-state distribution function $P_s(x,t)$ of the system can be written as

$$P_s(x,t) = \frac{N}{|D_{eff}(x,\tau,\lambda,t)|^{1/2}} \exp\left[-\frac{\Phi(x,\tau,\lambda,t)}{D}\right], \quad (21)$$

where the generalized potential $\Phi(x,\tau,\lambda,t)$ can be expressed as

$$\begin{aligned}\Phi(x,\tau,\lambda,t) = {} & \frac{\tau}{2R}x^4 - \frac{4\tau\lambda}{3R\sqrt{R}}x^3 + \alpha(\tau,\lambda)x^2 + \beta(\tau,\lambda)x \\ & + \gamma(\tau,\lambda)\ln|Rx^2 + 2\lambda\sqrt{R}x + 1| \\ & + \theta(\tau,\lambda)\arctan\frac{\sqrt{R}x+\lambda}{\sqrt{1-\lambda^2}} \\ & + \left(-\frac{\tau}{R}x + \frac{\lambda\tau}{R\sqrt{R}}\ln|Rx^2 + 2\lambda\sqrt{R}x + 1| \right. \\ & \left. - \mu(\tau,\lambda)\arctan\frac{\sqrt{R}x+\lambda}{\sqrt{1-\lambda^2}}\right) A\cos\Omega t \\ & + O(A^2) \quad \text{for } |\lambda| < 1, \end{aligned} \quad (22)$$

and

$$\begin{aligned}\Phi(x,\tau,\pm 1,t) = {} & \frac{\tau}{2R}x^4 \mp \frac{4\tau}{3R\sqrt{R}}x^3 + \alpha(\tau,\pm 1)x^2 + \beta(\tau,\pm 1)x \\ & + \gamma(\tau,\pm 1)\ln|Rx^2 \pm 2\sqrt{R}x + 1| \pm \frac{\kappa(\tau)}{\sqrt{R}x \pm 1} \\ & + \left(-\frac{\tau}{R}x \pm \frac{\tau}{R\sqrt{R}}\ln|Rx^2 \pm 2\sqrt{R}x + 1| \right. \\ & \left. + \frac{\rho(\tau)}{\sqrt{R}x \pm 1}\right) A\cos\Omega t + O(A^2) \quad \text{for } \lambda = \pm 1, \end{aligned} \quad (23)$$

with

$$\alpha(\tau,\lambda) = \frac{2\tau(4\lambda^2-1) - R(2\tau-1)}{2R^2}, \quad (24)$$

$$\beta(\tau,\lambda) = \frac{2\lambda[4\tau(1-2\lambda^2) + R(2\tau-1)]}{R^2\sqrt{R}}, \quad (25)$$

$$\gamma(\tau,\lambda) = \frac{2\tau(16\lambda^4 - 12\lambda^2 + 1) - R(2\tau-1)(4\lambda^2-1) - R^2}{2R^3}, \quad (26)$$

$$\theta(\tau,\lambda) = \frac{\lambda}{\sqrt{1-\lambda^2}} \times \frac{R^2 + R(2\tau-1)(4\lambda^2-3) - 2\tau(16\lambda^4 - 20\lambda^2 + 5)}{R^3}, \quad (27)$$

$$\mu(\tau,\lambda) = \frac{1}{\sqrt{1-\lambda^2}} \frac{\tau(2\lambda^2-1) + R(\tau+1)}{R\sqrt{R}}, \quad (28)$$

$$\kappa(\tau) = \frac{2\tau - R(2\tau-1) - R^2}{R^3}, \quad \rho(\tau) = \frac{\tau + R(\tau+1)}{R\sqrt{R}}. \quad (29)$$

From Eqs. (24)–(28), it can be seen that $\alpha(\tau,-\lambda) = \alpha(\tau,\lambda)$, $\beta(\tau,-\lambda) = -\beta(\tau,\lambda)$, $\gamma(\tau,-\lambda) = \gamma(\tau,\lambda)$, $\theta(\tau,-\lambda) = -\theta(\tau,\lambda)$, and $\mu(\tau,-\lambda) = \mu(\tau,\lambda)$.

The transition rate $W(x(t=0) = x_\pm, \tau, \lambda)$ out of the $x_\pm$ states can be approximately obtained from Eq. (14) since $W(x_\pm,\tau,\lambda) = \mathcal{T}_\pm^{-1}$,

$$W(x_\pm,\tau,|\lambda|<1)\simeq\frac{1}{\sqrt{2}\pi}\exp\Bigg(-\frac{1}{D}\Bigg[-\frac{\tau}{2R}\pm\frac{4\tau\lambda}{3R\sqrt{R}}-\alpha(\tau,\lambda)\mp\beta(\tau,\lambda)-\gamma(\tau,\lambda)\ln|R\pm2\lambda\sqrt{R}+1| -\theta(\tau,\lambda)\left(\arctan\frac{\lambda\pm\sqrt{R}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\pm\Bigg\{\frac{\tau}{R}\mp\frac{\tau\lambda}{R\sqrt{R}}\ln|R\pm2\lambda\sqrt{R}+1| \pm\mu(\tau,\lambda)\left(\arctan\frac{\lambda\pm\sqrt{R}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\Bigg\}A\cos\Omega t\Bigg]\Bigg), \tag{30}$$

$$W(x_\pm,\tau,\lambda=+1)\simeq\frac{1}{\sqrt{2}\pi}\exp\Bigg\{-\frac{1}{D}\Bigg[-\frac{\tau}{2R}\pm\frac{4\tau}{3R\sqrt{R}} -\alpha(\tau,+1)\mp\beta(\tau,+1)-\gamma(\tau,+1) \times\ln|R\pm2\sqrt{R}+1|\pm\frac{\sqrt{R}\kappa(\tau)}{1\pm\sqrt{R}} \pm\left(\frac{\tau}{R}\mp\frac{\tau}{R\sqrt{R}}\ln|R\pm2\sqrt{R}+1| +\frac{\sqrt{R}\rho(\tau)}{1\pm\sqrt{R}}\right)A\cos\Omega t\Bigg]\Bigg\}, \tag{31}$$

$$W(x_\pm,\tau,\lambda=-1)\simeq\frac{1}{\sqrt{2}\pi}\exp\Bigg\{-\frac{1}{D}\Bigg[-\frac{\tau}{2R}\mp\frac{4\tau}{3R\sqrt{R}} -\alpha(\tau,-1)\mp\beta(\tau,-1)-\gamma(\tau,-1) \times\ln|R\mp2\sqrt{R}+1|\mp\frac{\sqrt{R}\kappa(\tau)}{1\mp\sqrt{R}} \pm\left(\frac{\tau}{R}\pm\frac{\tau}{R\sqrt{R}}\ln|R\mp2\sqrt{R}+1| +\frac{\sqrt{R}\rho(\tau)}{1\mp\sqrt{R}}\right)A\cos\Omega t\Bigg]\Bigg\}. \tag{32}$$

The standard form of the signal-to-noise ratio R_{SNR} for the bistable system with correlations between noises in terms of the output signal power spectrum can be given by

$$R_{SNR}=\frac{\pi W_1^2(x_\pm,\lambda)}{4W_0(x_\pm,\lambda)}\left[1-\frac{W_1^2(x_\pm,\lambda)}{2[W_0^2(x_\pm,\lambda)+\Omega^2]}\right]^{-1}, \tag{33}$$

where $W_0(x_\pm,\lambda)$ and $W_1(x_\pm,\lambda)$ are the following.

(i) For $|\lambda|<1$,

$$W_0(x_\pm,\lambda)=\frac{\sqrt{2}}{\pi}\exp\Bigg\{-\frac{1}{D}\Bigg[-\frac{\tau}{2R}\pm\frac{4\tau\lambda}{3R\sqrt{R}}-\alpha(\tau,\lambda) \mp\beta(\tau,\lambda)-\gamma(\tau,\lambda)\ln|R\pm2\lambda\sqrt{R}+1|-\theta(\tau,\lambda) \times\left(\arctan\frac{\lambda\pm\sqrt{R}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\Bigg]\Bigg\}, \tag{34}$$

$$W_1(x_\pm,\lambda)=\frac{W_0(x_\pm,\lambda)A}{D}\Bigg[\frac{\tau}{R}\mp\frac{\tau\lambda}{R\sqrt{R}}\ln|R\pm2\lambda\sqrt{R}+1| \pm\mu(\tau,\lambda)\left(\arctan\frac{\lambda\pm\sqrt{R}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\Bigg]. \tag{35}$$

(ii) For $\lambda=+1$,

$$W_0(x_\pm,+1)=\frac{\sqrt{2}}{\pi}\exp\Bigg\{-\frac{1}{D}\Bigg[-\frac{\tau}{2R}\pm\frac{4\tau}{3R\sqrt{R}}-\alpha(\tau,+1) \mp\beta(\tau,+1)-\gamma(\tau,+1)\ln|R\pm2\sqrt{R}+1| \pm\frac{\kappa(\tau)\sqrt{R}}{1\pm\sqrt{R}}\Bigg]\Bigg\}, \tag{36}$$

$$W_1(x_\pm,+1)=\frac{W_0(x_\pm,+1)A}{D}\Bigg[\frac{\tau}{R}\mp\frac{\tau}{R\sqrt{R}}\ln|R\pm2\sqrt{R}+1| +\frac{\rho(\tau)\sqrt{R}}{1\pm\sqrt{R}}\Bigg]. \tag{37}$$

(iii) For $\lambda=-1$,

$$W_0(x_\pm,-1)=\frac{\sqrt{2}}{\pi}\exp\Bigg\{-\frac{1}{D}\Bigg[-\frac{\tau}{2R}\mp\frac{4\tau}{3R\sqrt{R}}-\alpha(\tau,-1) \mp\beta(\tau,-1)-\gamma(\tau,-1)\ln|R\mp2\sqrt{R}+1| \mp\frac{\kappa(\tau)\sqrt{R}}{1\mp\sqrt{R}}\Bigg]\Bigg\}, \tag{38}$$

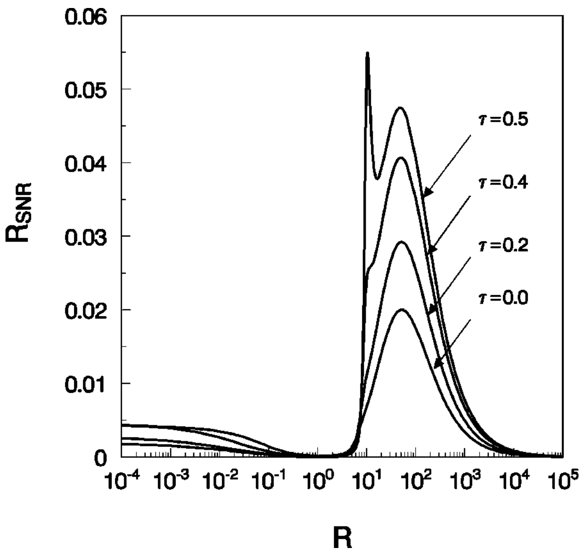

FIG. 2. SNR for the case of correlations between noises (the case of $|\lambda|<1$) with $\lambda=-0.7$ as a function of the noise intensities ratio R for different values of the correlation time τ of the colored noise. $A=0.05$, $\Omega=0.001$, and $D=0.05$.

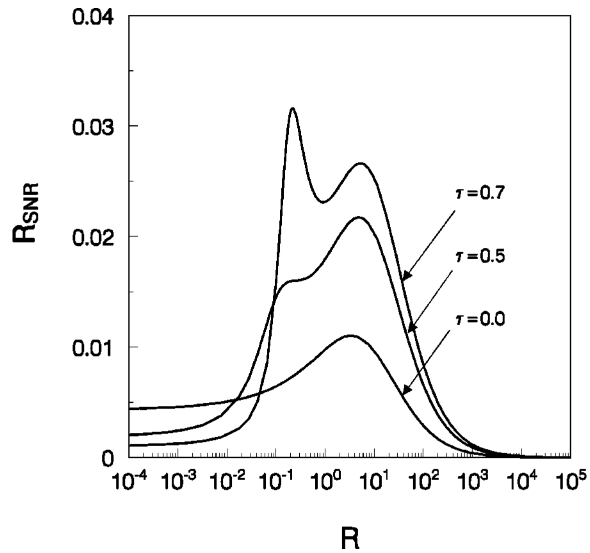

FIG. 3. SNR for the case of correlations between noises (the case of $|\lambda|<1$) with $\lambda=+0.7$ as a function of the noise intensities ratio R for different values of the correlation time τ of the colored noise. $A=0.05$, $\Omega=0.001$, and $D=0.05$.

$$W_1(x_\pm,-1)=\frac{W_0(x_\pm,-1)A}{D}\left[\frac{\tau}{R}\pm\frac{\tau}{R\sqrt{R}}\ln|R\mp2\sqrt{R}+1|+\frac{\rho(\tau)\sqrt{R}}{1\mp\sqrt{R}}\right]. \tag{39}$$

Although the effects of correlation between noises on the SR phenomenon have been discussed in Ref. [21], yet the multiplicative noise and the additive noise are all white noises there, and the effects of colored noise on SR have not been studied. On the other hand, because $\alpha(\tau,\lambda)$ and $\gamma(\tau,\lambda)$ are symmetric functions of the correlation intensity λ [i.e., $\alpha(\tau,-\lambda)=\alpha(\tau,\lambda)$, $\gamma(\tau,-\lambda)=\gamma(\tau,\lambda)$], and $\beta(\tau,\lambda)$ and $\theta(\tau,\lambda)$ are antisymmetric functions of the correlation intensity λ [i.e., $\beta(\tau,-\lambda)=-\beta(\tau,\lambda)$ and $\theta(\tau,-\lambda)=-\theta(\tau,\lambda)$], it can be found that (i) when $|\lambda|<1$, the SNR for $x(t=0)=x_+$ and $\lambda>0$ is equal to that for $x(t=0)=x_-$ and $\lambda<0$; the SNR for $x(t=0)=x_-$ and $\lambda>0$ is equal to that for $x(t=0)=x_+$ and $\lambda<0$. (ii) When $|\lambda|=1$, the SNR for $x(t=0)=x_+$ and $\lambda=+1$ is equal to that for $x(t=0)=x_-$ and $\lambda=-1$, and the SNR for $x(t=0)=x_-$ and $\lambda=+1$ is equal to that for $x(t=0)=x_+$ and $\lambda=-1$. Therefore, we can just discuss the effects of colored noise on the SNR for the following four cases: the case of $x(t=0)=x_+$ and $\lambda<0$, the case of $x(t=0)=x_+$ and $\lambda>0$, the case of $x(t=0)=x_+$ and $\lambda=+1$, and the case of $x(t=0)=x_-$ and $\lambda=+1$ since there are some inherent symmetries on the SNR as mentioned above.

For the case of $x(t=0)=x_+$ and $\lambda<0$ [or $x(t=0)=x_-$ and $\lambda>0$], we present the SNR as a function of the noise intensities ratio $R\equiv Q/D$ for different values of correlation time τ of the colored noise in Fig. 2. It is found that, when $R>1$ (or $Q>D$), there is only one peak in the SNR (i.e., the SSR phenomenon) for a small value of correlation time τ, and the DSR phenomenon (i.e., there are two peaks in the SNR) will appear as τ increases. It should be pointed out that the two peaks are located in the region of $R>1$. When $R<1$ (or $Q<D$), the SNR increases but saturates to a plateau value as R decreases, and there is no peak in this region. However, for the case of $x(t=0)=x_+$ and $\lambda>0$ [or $x(t=0)=x_+$ and $\lambda<0$], it is shown in Fig. 3 that there is one peak in the SNR in the region of $R>1$, and a second peak will appear in the region of $R<1$ as the correlation time of colored noise is increased. When $R\to0$, the SNR decreases but saturates to a plateau value, and the signal-to-noise ratio

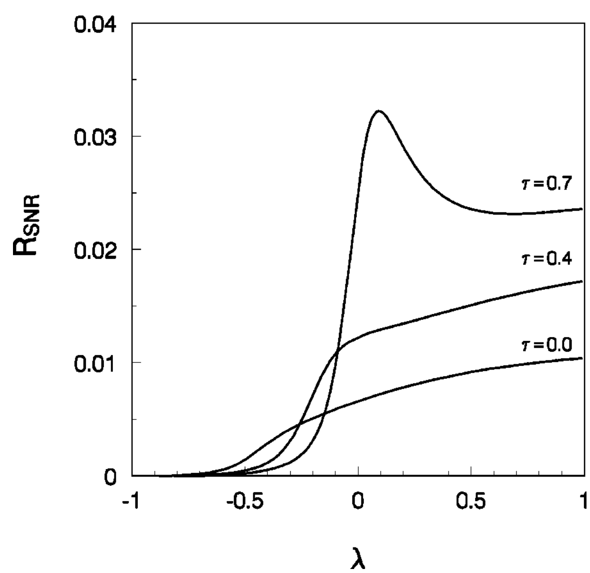

FIG. 4. When the noise intensities ratio R is fixed, $R=1$; SNR for the case of correlations between noises (the case of $|\lambda|<1$) as a function of the correlative intensity λ for different values of the correlation time τ of the colored noise. $A=0.05$, $\Omega=0.001$, and $D=0.05$.

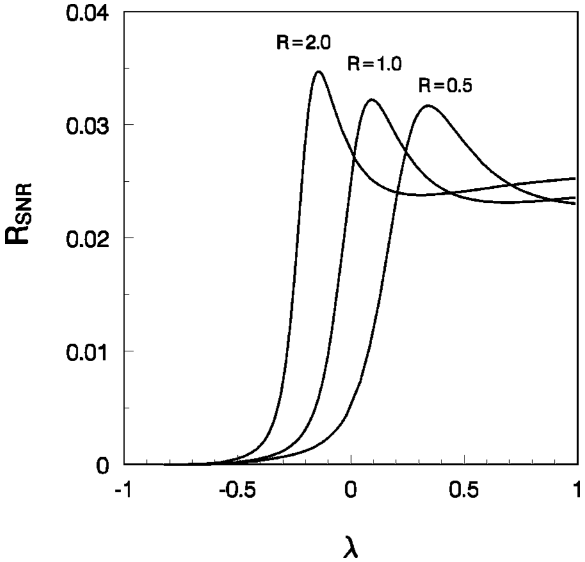

FIG. 5. When the correlation time τ of the colored noise is fixed, $\tau=0.7$; SNR for the case of correlations between noises (the case of $|\lambda|<1$) as a function of the correlative intensity λ for different values of the noise intensities ratio R. $A=0.05$, $\Omega=0.001$, and $D=0.05$.

vanishes when $R\to\infty$. In Figs. 4 and 5 we present the SNR as a function of noise correlative intensity λ for the different values of correlation time τ and for the different values of noise intensities ratio R, respectively. There is a peak for a large value of correlation time (e.g., $\tau=0.7$ in Fig. 4), it means that there is an optimal correlative intensity at which the SR phenomenon can occur. When correlation time is fixed (e.g., see Fig. 5, $\tau=0.7$), our computation shows that the SR phenomenon can occur for a different value of R, and the SNR decreases but saturates to a plateau value as $\lambda\to+1$.

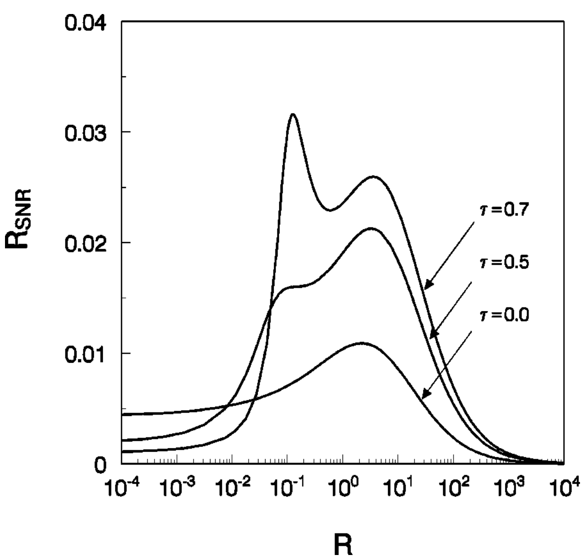

FIG. 6. SNR for the case of correlations between noises [the case of $\lambda=+1$ and $x(t=0)=x_+$] as a function of the noise intensities ratio R for different values of the correlation time τ of the colored noise. $A=0.05$, $\Omega=0.001$, and $D=0.05$.

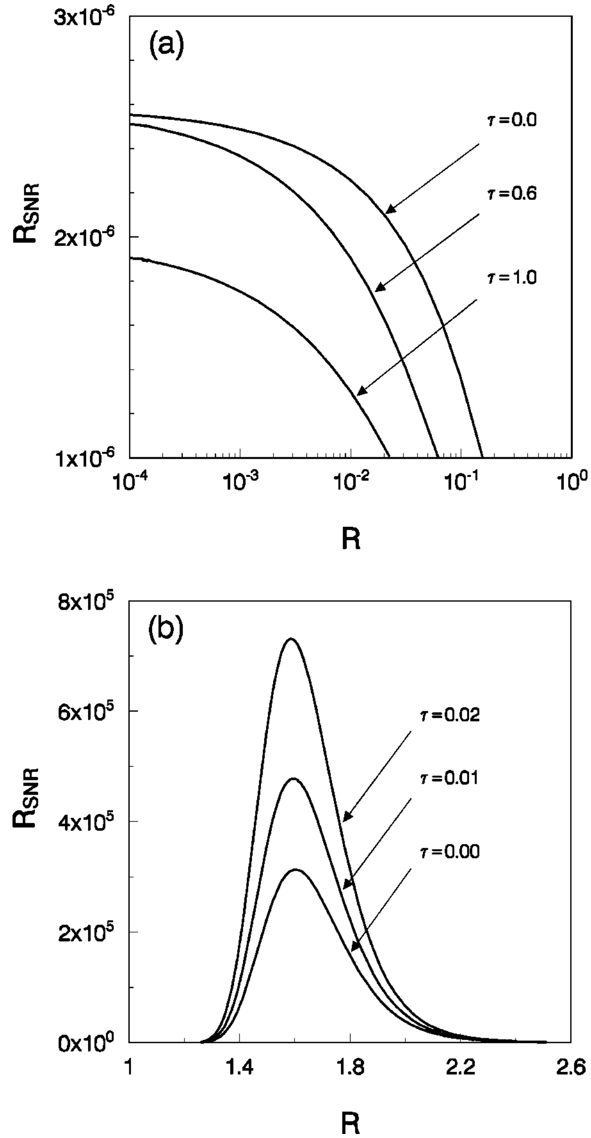

FIG. 7. SNR for the case of correlations between noises [the case of $\lambda=+1$ and $x(t=0)=x_-$] as a function of the noise intensities ratio R for different values of the correlation time τ of the colored noise. $A=0.001$, $\Omega=0.001$, and $D=0.085$. (a) $R<1$ or $Q<D$, (b) $R>1$ or $Q>D$.

For the case of $x(t=0)=x_+$ and $\lambda=+1$ [or $x(t=0)=x_-$ and $\lambda=-1$], in Fig. 6 we present the SNR as a function of the noise intensities ratio R for different values of correlation time τ. When $\tau=0$ (the case of white noise) and τ is small, there is one peak that is located in the region of $R>1$. As the correlation time τ is increased, a second peak appears in the region of $R<1$, and the DSR appears as the correlation time increases. The SNR decreases but saturates to a plateau value as $R\to0$, and vanishes as $R\to\infty$. For the case of $x(t=0)=x_-$ and $\lambda=+1$ [or $x(t=0)=x_+$ and $\lambda=+1$], the SNR as a function of the noise intensities ratio R for different values of correlation time τ is presented in Fig. 7. When $R<1$, our computation shows that there is no peak in the SNR [see Fig. 7(a)], it increases but saturates to a plateau value as $R\to0$, the value of the SNR is very small (of the order of 10^{-6}) and it decreases as the correlation time τ is increased. However, when $R>1$ there is one peak for different τ values over a very narrow range in R [see Fig. 7(b)], the value of the SNR is very large (of the order of 10^5) and the SNR increases as the correlation time τ is increased. There is a large variation in the value of the SNR peak over a very narrow range in τ. Moreover, it should be pointed out that there is no DSR phenomenon for $R>1$.

IV. CONCLUSIONS

In this paper, we have discussed the effects of colored noise on the SR in conventional bistable systems by using the theory of SNR [9]. First of all, the general equations of nonlinear systems under the simultaneous action of multiplicative colored noise and additive white noise are derived by applying the unified colored noise approximation [25]. Second, considering the conventional bistable system addition of the action of a periodic signal, we study the SR phenomenon in the bistable system and two cases have been considered: one is the case of no correlations between noises and the other is the case of correlations between noises. The expressions of the SNR for both cases have been obtained. Third, the effects of the colored noise on the SR phenomenon have been discussed through numerical computation. It is found that, in the case of no correlations between noises, there is an optimal noise intensities ratio R at which SNR is maximum (one peak) that identifies a characteristic of the SR phenomenon when the correlation time τ is small. However, when the correlation time is increased, a second optimal value of the noise intensities ratio appears, i.e., there are two peaks in the SNR that has been called *double stochastic resonance* (i.e., DSR). Although a similar DSR phenomenon has been shown in Refs. [9] and [21], yet this phenomenon appears for the increasing correlation time τ of the multiplicative colored noise here.

In the case of correlations between noises, the SNR is not only dependent on the correlation time of the colored noise, but also on the intensity λ of correlations between noises. When $|\lambda|<1$ and $\lambda=+1$ with $x(t=0)=x_+$ [or $\lambda=-1$ with $x(t=0)=x_-$], the DSR appears as the correlation time is increased. However, when $\lambda=+1$ with $x(t=0)=x_-$ [or $\lambda=-1$ with $x(t=0)=x_+$], there is no DSR phenomenon. When the noise intensities ratio is fixed, there is an optimal correlative intensity where the SSR occurs. When the correlation time of colored noise is fixed, the SSR phenomenon can occur for different values of the noise intensities ratio.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 19805003 and the Fok Ying Tung Education Foundation under Grant No. 71007.

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